Journal of Applied Mechanics and Technical Physics, Vol. 43, No. 4, pp. 544–551, 2002

MODEL OF A TURBULENT BOUNDARY LAYER WITH EXPLICIT IDENTIFICATION OF THE COHERENT GENERATION STRUCTURE

V. V. Bogolepov,[1] **V. A. Zharov,**[1]
I. I. Lipatov,[1] **and Yu. I. Khlopkov**[2]

UDC 532.517.4:533.9

Using the method of moments in the space of wavenumbers, a class of models of a developed turbulent flow of an incompressible fluid in a flat-plate boundary layer is proposed. The models are based on an analysis of the Navier–Stokes equations that describe the behavior of dynamic coherent structures associated with vorticity generation and also the behavior of the stochastic component. A continuum analog of dynamic equations for a coherent structure is given in an explicit form. In the general case, the stochastic component should satisfy a system of equations of the kinetic type, which reduces to one equation under certain assumptions. It is also shown that the presence of coherent structures leads to generalization of the notion of statistical homogeneity.

Introduction. The difficulties in describing turbulent motion of a fluid is caused, among other reasons, by the fact that the spectrum of fluctuations is continuous. Two methods are currently used to describe turbulent motion of a fluid. In one method proposed by Boussinesq [1] and Prandtl [2], a linear relationship is established between the stress tensor and strain-rate tensor, and the mixing-length theory is used for the proportionality coefficient. The spectrum continuity is associated with the dependence of the mixing length on coordinates (for instance, the mixing length in the boundary layer is proportional to the distance from the wall). Formulation of the problem, for example, about the boundary layer is essentially stationary; in addition, this form of the stress tensor is within the framework of physical boundary conditions.

The other method for describing turbulent motion was proposed by Smagorinsky [3] (see also [4]). In this formulation, the problem of a turbulent flow around a body is solved numerically. In accordance with the turbulent flow structure used, the grid of the numerical scheme is chosen such that the wavenumber based on the grid size is within the inertial region. Then, all quantities that have the size of the grid cell or smaller can be averaged, based on the known results for uniform and isotropic turbulence, which leads to an effective (so-called subgrid) stress tensor with a certain (according to Smagorinsky, linear) relationship with the strain-rate tensor. It should be noted that the formulation of the problem is nonstationary for the long-wave part of the spectrum. The Reynolds equations are used as initial ones, and their solution is supplemented by the operation of filtration of the low-frequency part. Thus, this formulation is actually related to identification of the long-wave component of motion, i.e., it is an asymptotic solution. However, problem formulation includes inconsistency from the viewpoint of asymptotic methods.

The model with the linear relationship of the stress tensor does not describe some physical effects that are definitely established at the moment [5]: reverse flux of energy over the spectrum, anisotropy of the stress tensor, and the presence of coherent structures. Advanced methods [4] allow one to obtain a more complicated, nonlinear, relationship between the stress tensor and the strain-rate tensor. The physical content of the theory is refined, but additional difficulties are involved: the stress tensor contains derivatives that increase the order of initial equations, which requires introduction of additional boundary conditions.

---

[1]Central Aerohydrodynamic Institute, Zhukovskii 140180. [2]Moscow Physicotechnical Institute, Zhukovskii 140180. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 43, No. 4, pp. 65–74, July–August, 2002. Original article submitted December 11, 2001.

 0021-8944/02/4304-0544 $27.00

The description of the short-wave part of the spectrum also involves some uncertainties. Advanced methods for studying turbulent flows operate with some random forces whose spectrum is set beforehand. Physically, however, this force is associated with thermal fluctuations in aninhomogeneous gas flow [6]. The problem of determining the characteristics of such fluctuations has not been solved yet even in the linear case in the boundary layer. Moreover, there are two approaches to solving this problem, which may be called the Langevinian (stochastic) and Liouvillian (dynamic) approaches.

Nevertheless, in formulation of the problem of turbulent fluctuations, the fluctuations, despite their small amplitude, are a short-wave asymptotic solution and, in a certain sense, are boundary conditions for the limiting transition to high wavenumbers.

In the problem of uniform and isotropic turbulence, the long-wave component (coherent structure) is set artificially. Physically, a coherent structure in a turbulent boundary layer is a certain process of vorticity generation [7, 8] described by a certain system of equations or a particular relation for the velocity-field components, which is the boundary condition for low wavenumbers.

Thus, we believe that a correct solution of the problem of turbulent fluctuations requires the knowledge of the long-wave (coherent structure) and short-wave components of turbulent fluctuations. The solution in the interval between high and low wavenumbers should satisfy these boundary conditions. Using the advanced method of analysis of such systems (for instance, the method of the recursive renormalization group [4]) one can try to solve the problem completely if these two boundary conditions are available. Nevertheless, neither the long-wave nor, moreover, the short-wave asymptotic solutions are currently known. It seems that a certain formulation of the problem, if possible at all, should be eclectic.

In the present paper, we made an attempt to determine the coherent component in a turbulent boundary layer on a flat plate using the method of moments [9].

1. Formulation of the Problem. We consider the possibility of constructing a certain set of models for a turbulent flow field in the boundary layer in a low-mode [10] approximation (in the simplest case, one-mode approximation). One of the special features of the turbulent boundary layer is the presence of a small parameter ε equal in order of magnitude to the square root of the dimensionless growth rate of the Tollmien–Schlichting waves (in this case, the Reynolds number, which is high by definition, is bounded by the relation $\varepsilon^2\mathrm{R} \gg 1$). The calculations of Dodonov et al. [11] show that all the modes of the Orr–Sommerfeld equation are stable on the velocity profile of the turbulent boundary layer. Terms of the third order of amplitude can change this situation due to addition of a generation term proportional to the integral intensity of fluctuations into the equations for fluctuations. However, the bursting observed in the experiment (periodic dynamic process of vorticity generation near the wall), low modes of the field of turbulent fluctuations found in calculations, which allow three-wave resonance, and also physical considerations (the presence of a mechanism of energy transfer from the mean flow to turbulent fluctuations) indicate that generation of energy fluctuations is most probably performed due to the coherent structure, which itself is a nonstationary vortex structure.

Thus, we consider a variant of the boundary-layer description, which contains nonstationary dynamic coherent structures in addition to stochastic fluctuations. We use the method of moments in the space of wavenumbers [9]. The method of many scales is used to solve the thus-obtained equations.

The velocity field is divided into two components: mean-time and fluctuating. The equations for the coherent component are separated from the equations for random fluctuations in an explicit form due to the three-wave resonance of the Tollmien–Schlichting waves of the discrete spectrum, i.e., finally we have triple decomposition [8] of the velocity field of the turbulent flow. For the incoherent component, in the general case, we obtain a system of kinetic equations, which can be reduced to one equation under certain assumptions.

Following [10], we consider a developed turbulent boundary layer (Fig. 1). The initial equations that are assumed to describe the nonstationary flow field (Navier–Stokes equations) can be decomposed into the equations for the mean field U, V (in this paper, in the boundary-layer approximation) and the fluctuating field:

$$u = U + \varepsilon u', \qquad v = \varepsilon^2 V + \varepsilon v', \qquad w = \varepsilon w'.$$

The equations for fluctuations are reduced to a system, which includes the Orr–Sommerfeld equation for the vertical component of velocity and the Squire equation for the vertical component of vorticity [12]. Since decomposition in eigenfunctions of the linear parts of these equations possesses completeness [13], the solution of the problem can be represented as series in eigenfunctions.

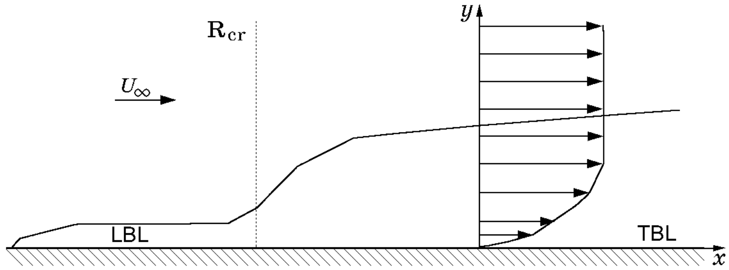

Fig. 1. Spatial configuration of the flow field in the boundary layer (the laminar and turbulent boundary layers are denoted as LBL and TBL, respectively).

2. Equations for Amplitudes in the One-Mode Approximation. We expand the vertical components of velocity and vorticity in eigenfunctions of the Orr–Sommerfeld and Squire equations, respectively:

$$\hat{v}_{\boldsymbol{k}}(y) = \sum_{n=0}^{N} A_{\boldsymbol{k}}^{(n)} \varphi_{\boldsymbol{k}}^{(n)}(y) + \int_{-\infty}^{\infty} A_{\boldsymbol{k}}^{(\mu)} \varphi_{\boldsymbol{k}}^{(\mu)}(y)\, d\mu, \quad \hat{\eta}_{\boldsymbol{k}}(y) = \sum_{n=0}^{N} B_{\boldsymbol{k}}^{(n)} \psi_{\boldsymbol{k}}^{(n)}(y) + \int_{-\infty}^{\infty} B_{\boldsymbol{k}}^{(\mu)} \psi_{\boldsymbol{k}}^{(\mu)}(y)\, d\mu.$$

Here $\hat{v}_{\boldsymbol{k}}$ and $\hat{\eta}_{\boldsymbol{k}}$ are the Fourier images of the vertical components of velocity and vorticity, summation is performed by modes (eigenfunctions of the Orr–Sommerfeld and Squire operators) of the discrete spectrum, and integration is performed by modes of the continuous spectrum. In the one-mode approximation, we obtain the following system (for simplicity, we consider only one unstable mode of the discrete spectrum, since modes of higher orders have large decrements):

$$\hat{v}_{\boldsymbol{k}}(y) = A_{\boldsymbol{k}} \varphi_{\boldsymbol{k}}^{(0)}(y) + \ldots, \qquad \hat{\eta}_{\boldsymbol{k}}(y) = B_{\boldsymbol{k}} \psi_{\boldsymbol{k}}^{(0)}(y) + \ldots,$$

$$\hat{u}_{\boldsymbol{k}}(y) = \frac{i}{k^2}\Big(\alpha\, \frac{d\varphi_{\boldsymbol{k}}^{(0)}(y)}{dy}\, A_{\boldsymbol{k}} - \beta \psi_{\boldsymbol{k}}^{(0)}(y) B_{\boldsymbol{k}}\Big) + \ldots,$$

$$\hat{w}_{\boldsymbol{k}}(y) = \frac{i}{k^2}\Big(\beta\, \frac{d\varphi_{\boldsymbol{k}}^{(0)}(y)}{dy}\, A_{\boldsymbol{k}} + \alpha \psi_{\boldsymbol{k}}^{(0)}(y) B_{\boldsymbol{k}}\Big) + \ldots\ .$$

After substitution of the Tollmien–Schlichting wave frequencies

$$A_{\boldsymbol{k}} = \bar{A}_{\boldsymbol{k}} \exp(-i \operatorname{Re}[\omega_{OS}^{(0)}]t), \qquad B_{\boldsymbol{k}} = \bar{B}_{\boldsymbol{k}} \exp(-i \operatorname{Re}[\omega_{OS}^{(0)}]t),$$

the equations for the amplitudes $\bar{A}_{\boldsymbol{k}}$ and $\bar{B}_{\boldsymbol{k}}$ acquire the following form:

$$-\frac{\partial}{\partial t}\, \bar{A}_{\boldsymbol{k}} + \varepsilon^2 \operatorname{Im}[\omega_{OS}^{(0)}(\boldsymbol{k})] A_{\boldsymbol{k}} = \varepsilon \int \bar{H}_{\boldsymbol{k}\boldsymbol{k}_1\boldsymbol{k}_2}^{(1)} \bar{A}_{\boldsymbol{k}_1} \bar{A}_{\boldsymbol{k}_2}\, d\boldsymbol{k}_1 + \varepsilon \int \bar{H}_{\boldsymbol{k}\boldsymbol{k}_1\boldsymbol{k}_2}^{(2)} \bar{A}_{\boldsymbol{k}_1} \bar{B}_{\boldsymbol{k}_2}\, d\boldsymbol{k}_1$$

$$+ \varepsilon \int \bar{H}_{\boldsymbol{k}\boldsymbol{k}_1\boldsymbol{k}_2}^{(3)} \bar{B}_{\boldsymbol{k}_1} \bar{B}_{\boldsymbol{k}_2}\, d\boldsymbol{k}_1 - \varepsilon^2 h_{\boldsymbol{k}}^{(1)} \bar{A}_{\boldsymbol{k}} - \varepsilon^2 h_{\boldsymbol{k}}^{(2)} \bar{B}_{\boldsymbol{k}}; \tag{2.1}$$

$$\frac{\partial}{\partial t}\, \bar{B}_{\boldsymbol{k}} = \varepsilon \int \bar{G}_{\boldsymbol{k}\boldsymbol{k}_1\boldsymbol{k}_2}^{(1)} \bar{A}_{\boldsymbol{k}_1} \bar{A}_{\boldsymbol{k}_2}\, d\boldsymbol{k}_1 + \varepsilon \int \bar{G}_{\boldsymbol{k}\boldsymbol{k}_1\boldsymbol{k}_2}^{(2)} \bar{A}_{\boldsymbol{k}_1} \bar{B}_{\boldsymbol{k}_2}\, d\boldsymbol{k}_1 + \varepsilon \int \bar{G}_{\boldsymbol{k}\boldsymbol{k}_1\boldsymbol{k}_2}^{(3)} \bar{B}_{\boldsymbol{k}_1} \bar{B}_{\boldsymbol{k}_2}\, d\boldsymbol{k}_1$$

$$-\, \varepsilon^2 g_{\boldsymbol{k}}^{(1)} \bar{A}_{\boldsymbol{k}} - \varepsilon^2 g_{\boldsymbol{k}}^{(2)} \bar{B}_{\boldsymbol{k}} - i\beta N_S\Big(\psi_{\boldsymbol{k}}^{(0)}, \frac{d\varphi_{\boldsymbol{k}}^{(0)}}{dy}\Big) \bar{A}_{\boldsymbol{k}} + i(\operatorname{Re}[\omega_{OS}^{(0)}(\boldsymbol{k})] - \omega_S^{(0)}(\boldsymbol{k})) \bar{B}_{\boldsymbol{k}}; \tag{2.2}$$

$$\bar{H}_{\boldsymbol{k}\boldsymbol{k}_1\boldsymbol{k}_2}^{(i)} = H_{\boldsymbol{k}\boldsymbol{k}_1\boldsymbol{k}_2}^{(i)} \exp[i \operatorname{Re}(\omega_{OS}^{(0)}(\boldsymbol{k}) - \omega_{OS}^{(0)}(\boldsymbol{k}_1) - \omega_{OS}^{(0)}(\boldsymbol{k}_2))t],$$

$$\bar{G}_{\boldsymbol{k}\boldsymbol{k}_1\boldsymbol{k}_2}^{(i)} = G_{\boldsymbol{k}\boldsymbol{k}_1\boldsymbol{k}_2}^{(i)} \exp[i \operatorname{Re}(\omega_{OS}^{(0)}(\boldsymbol{k}) - \omega_{OS}^{(0)}(\boldsymbol{k}_1) - \omega_{OS}^{(0)}(\boldsymbol{k}_2))t].$$

Here $\boldsymbol{k}_2 = \boldsymbol{k} - \boldsymbol{k}_1$, $\varepsilon = d/L$, d is the momentum-loss length, L is the characteristic longitudinal scale, the quantities $H_{\boldsymbol{k}\boldsymbol{k}_1\boldsymbol{k}_2}^{(i)}$ and $G_{\boldsymbol{k}\boldsymbol{k}_1\boldsymbol{k}_2}^{(i)}$ are symmetric in the case of the permutation $\boldsymbol{k}_1 \leftrightarrow \boldsymbol{k}_2$, and $N_S(f,g) = \displaystyle\int_0^\infty f(y) g(y)\, dy$.

3. Equations for Amplitudes with Accuracy to $O(\varepsilon^2)$. Assuming that the decrement of the lowest mode of the Squire equation is high, i.e., this mode is continuously "adapted" to the Tollmien–Schlichting mode, we can obtain the amplitude of the vertical component of vorticity in the form

$$\bar{B}_{\boldsymbol{k}} = \frac{\beta N_S(\psi^{(0)}_{\boldsymbol{k}}, d\varphi^{(0)}_{\boldsymbol{k}}/dy)\bar{A}_{\boldsymbol{k}}}{\mathrm{Re}(\omega^{(0)}_{OS}(\boldsymbol{k})) - \omega^{(0)}_S(\boldsymbol{k})} - \frac{\varepsilon}{i(\mathrm{Re}(\omega^{(0)}_{OS}(\boldsymbol{k})) - \omega^{(0)}_S(\boldsymbol{k}))}$$

$$\times \Big(\int \bar{G}^{(1)}_{\boldsymbol{k}\boldsymbol{k}_1\boldsymbol{k}_2}\bar{A}_{\boldsymbol{k}_1}\bar{A}_{\boldsymbol{k}_2}\,d\boldsymbol{k}_1 + \int \bar{G}^{(2)}_{\boldsymbol{k}\boldsymbol{k}_1\boldsymbol{k}_2}\,\frac{\beta_2 N_S(\psi^{(0)}_{\boldsymbol{k}_2}, d\varphi^{(0)}_{\boldsymbol{k}_2}/dy)}{\mathrm{Re}\,(\omega^{(0)}_{OS}(\boldsymbol{k}_2)) - \omega^{(0)}_S(\boldsymbol{k}_2)}\,\bar{A}_{\boldsymbol{k}_1}\,\bar{A}_{\boldsymbol{k}_2}\,d\boldsymbol{k}_1$$

$$+ \int \bar{G}^{(3)}_{\boldsymbol{k}\boldsymbol{k}_1\boldsymbol{k}_2}\,\frac{\beta_1 N_S(\psi^{(0)}_{\boldsymbol{k}_1}, d\varphi^{(0)}_{\boldsymbol{k}_1}/dy)}{\mathrm{Re}\,(\omega^{(0)}_{OS}(\boldsymbol{k}_1)) - \omega^{(0)}_S(\boldsymbol{k}_1)}\,\frac{\beta_2 N_S(\psi^{(0)}_{\boldsymbol{k}_2}, d\varphi^{(0)}_{\boldsymbol{k}_2}/dy)}{\mathrm{Re}\,(\omega^{(0)}_{OS}(\boldsymbol{k}_2)) - \omega^{(0)}_S(\boldsymbol{k}_2)}\,\bar{A}_{\boldsymbol{k}_1}\bar{A}_{\boldsymbol{k}_2}\,d\boldsymbol{k}_1\Big) + O(\varepsilon^2) \quad (\boldsymbol{k}_2 = \boldsymbol{k} - \boldsymbol{k}_1)$$

or, in a shorter form,

$$\bar{B}_{\boldsymbol{k}} = b_{\boldsymbol{k}}\bar{A}_{\boldsymbol{k}} - \varepsilon \int G_{\boldsymbol{k}\boldsymbol{k}_1\boldsymbol{k}_2}\bar{A}_{\boldsymbol{k}_1}\bar{A}_{\boldsymbol{k}_2}d\boldsymbol{k}_1 + O(\varepsilon^2), \qquad b_{\boldsymbol{k}} = \frac{\beta N_S(\psi^{(0)}_{\boldsymbol{k}}, d\varphi^{(0)}_{\boldsymbol{k}}/dy)}{\mathrm{Re}\,(\omega^{(0)}_{OS}(\boldsymbol{k})) - \omega^{(0)}_S(\boldsymbol{k})}, \tag{3.1}$$

$$G_{\boldsymbol{k}\boldsymbol{k}_1\boldsymbol{k}_2} = \frac{\bar{G}^{(1)}_{\boldsymbol{k}\boldsymbol{k}_1\boldsymbol{k}_2}}{i(\mathrm{Re}\,(\omega^{(0)}_{OS}(\boldsymbol{k})) - \omega^{(0)}_S(\boldsymbol{k}))} + \frac{\bar{G}^{(2)}_{\boldsymbol{k}\boldsymbol{k}_1\boldsymbol{k}_2}}{i(\mathrm{Re}\,(\omega^{(0)}_{OS}(\boldsymbol{k})) - \omega^{(0)}_S(\boldsymbol{k}))}\,\frac{\beta_2 N_S(\psi^{(0)}_{\boldsymbol{k}_2}, d\varphi^{(0)}_{\boldsymbol{k}_2}/dy)}{\mathrm{Re}\,(\omega^{(0)}_{OS}(\boldsymbol{k}_2)) - \omega^{(0)}_S(\boldsymbol{k}_2)}$$

$$+ \frac{\bar{G}^{(3)}_{\boldsymbol{k}\boldsymbol{k}_1\boldsymbol{k}_2}}{i(\mathrm{Re}\,(\omega^{(0)}_{OS}(\boldsymbol{k})) - \omega^{(0)}_S(\boldsymbol{k}))}\,\frac{\beta_1 N_S(\psi^{(0)}_{\boldsymbol{k}_1}, d\varphi^{(0)}_{\boldsymbol{k}_1}/dy)}{\mathrm{Re}\,(\omega^{(0)}_{OS}(\boldsymbol{k}_1)) - \omega^{(0)}_S(\boldsymbol{k}_1)}\,\frac{\beta_2 N_S(\psi^{(0)}_{\boldsymbol{k}_2}, d\varphi^{(0)}_{\boldsymbol{k}_2}/dy)}{\mathrm{Re}\,(\omega^{(0)}_{OS}(\boldsymbol{k}_2)) - \omega^{(0)}_S(\boldsymbol{k}_2)} \quad (\boldsymbol{k}_2 = \boldsymbol{k} - \boldsymbol{k}_1).$$

The values of $\bar{B}_{\boldsymbol{k}}$ from (3.1) can be substituted into Eq. (2.1) for $\bar{A}_{\boldsymbol{k}}$. Finally, we obtain

$$-\frac{\partial \bar{A}_{\boldsymbol{k}}}{\partial t} + \varepsilon^2\,\mathrm{Im}\,(\bar{\omega}^{(0)}_{OS}(\boldsymbol{k}))\bar{A}_{\boldsymbol{k}} = \varepsilon \int H_{\boldsymbol{k}\boldsymbol{k}_1\boldsymbol{k}_2}\bar{A}_{\boldsymbol{k}_1}\bar{A}_{\boldsymbol{k}-\boldsymbol{k}_1}\,d\boldsymbol{k}_1 - \varepsilon^2 \int F_{\boldsymbol{k}\boldsymbol{k}_1\boldsymbol{k}_2}\bar{A}_{\boldsymbol{k}_1}\bar{A}_{\boldsymbol{k}_2}\bar{A}_{\boldsymbol{k}-\boldsymbol{k}_1-\boldsymbol{k}_2}\,d\boldsymbol{k}_1\,d\boldsymbol{k}_2 - \varepsilon^2 h_{\boldsymbol{k}}\bar{A}_{\boldsymbol{k}},$$

$$\varepsilon^2 = \max_{\boldsymbol{k}}\,[\mathrm{Im}\,(\omega^{(0)}_{OS}(\boldsymbol{k}))] = d/L, \qquad \mathrm{Im}\,(\bar{\omega}^{(0)}_{OS}(\boldsymbol{k})) = \mathrm{Im}\,(\omega^{(0)}_{OS}(\boldsymbol{k}))/\max_{\boldsymbol{k}}\,[\mathrm{Im}\,(\omega^{(0)}_{OS}(\boldsymbol{k}))].$$

4. Separation of Fluctuations into Coherent and Incoherent Components. After the above transformations for the amplitudes of the Tollmien–Schlichting waves, we obtain the following equation:

$$\frac{\partial \bar{A}_{\boldsymbol{k}}}{\partial t} = -\varepsilon \int H_{\boldsymbol{k}\boldsymbol{k}_1\boldsymbol{k}_2}\bar{A}_{\boldsymbol{k}_1}\bar{A}_{\boldsymbol{k}-\boldsymbol{k}_1}\,d\boldsymbol{k}_1 + \varepsilon^2\Big(\Omega_{\boldsymbol{k}}\bar{A}_{\boldsymbol{k}} + \int F_{\boldsymbol{k}\boldsymbol{k}_1\boldsymbol{k}_2}\bar{A}_{\boldsymbol{k}_1}\bar{A}_{\boldsymbol{k}_2}\bar{A}_{\boldsymbol{k}-\boldsymbol{k}_1-\boldsymbol{k}_2}\,d\boldsymbol{k}_1\,d\boldsymbol{k}_2\Big),$$

$$\Omega_{\boldsymbol{k}} = \mathrm{Im}\,(\bar{\omega}^{(0)}_{OS}(\boldsymbol{k})) + h_{\boldsymbol{k}}.$$

We express $\bar{A}_{\boldsymbol{k}}$ in the form

$$\bar{A}_{\boldsymbol{k}} = A^c_{\boldsymbol{k}} + A'_{\boldsymbol{k}}, \quad \langle A_{\boldsymbol{k}}\rangle = A^c_{\boldsymbol{k}}, \quad \langle A'_{\boldsymbol{k}}\rangle = 0$$

(i.e., divide into the coherent and incoherent components); the broken brackets denote averaging over random phases of fluctuations. We express $\bar{A}_{\boldsymbol{k}_1}\bar{A}_{\boldsymbol{k}_2}$ and $\bar{A}_{\boldsymbol{k}_1}\bar{A}_{\boldsymbol{k}_2}\bar{A}_{\boldsymbol{k}_3}$ through $A^c_{\boldsymbol{k}}$ and $A'_{\boldsymbol{k}}$:

$$\bar{A}_{\boldsymbol{k}_1}\bar{A}_{\boldsymbol{k}_2} = A^c_{\boldsymbol{k}_1}A^c_{\boldsymbol{k}_2} + A^c_{\boldsymbol{k}_1}A'_{\boldsymbol{k}_2} + A'_{\boldsymbol{k}_1}A^c_{\boldsymbol{k}_2} + A'_{\boldsymbol{k}_1}A'_{\boldsymbol{k}_2},$$

$$\bar{A}_{\boldsymbol{k}_1}\bar{A}_{\boldsymbol{k}_2}\bar{A}_{\boldsymbol{k}_3} = A^c_{\boldsymbol{k}_1}A^c_{\boldsymbol{k}_2}A^c_{\boldsymbol{k}_3} + A^c_{\boldsymbol{k}_1}A'_{\boldsymbol{k}_2}A^c_{\boldsymbol{k}_3} + A'_{\boldsymbol{k}_1}A^c_{\boldsymbol{k}_2}A^c_{\boldsymbol{k}_3} + A^c_{\boldsymbol{k}_1}A^c_{\boldsymbol{k}_2}A'_{\boldsymbol{k}_3} \tag{4.1}$$

$$+ A'_{\boldsymbol{k}_1}A'_{\boldsymbol{k}_2}A^c_{\boldsymbol{k}_3} + A^c_{\boldsymbol{k}_1}A'_{\boldsymbol{k}_2}A'_{\boldsymbol{k}_3} + A'_{\boldsymbol{k}_1}A^c_{\boldsymbol{k}_2}A'_{\boldsymbol{k}_3} + A'_{\boldsymbol{k}_1}A'_{\boldsymbol{k}_2}A'_{\boldsymbol{k}_3}.$$

We average the equation for fluctuations over an ensemble of random phases. For the coherent component, with allowance for Eq. (4.1), we obtain

$$\frac{\partial A^c_{\boldsymbol{k}}}{\partial t} = -\varepsilon \int H_{\boldsymbol{k}\boldsymbol{k}_1\boldsymbol{k}_2}(A^c_{\boldsymbol{k}_1}A^c_{\boldsymbol{k}_2=\boldsymbol{k}-\boldsymbol{k}_1} + \langle A'_{\boldsymbol{k}_1}A'_{\boldsymbol{k}_2}\rangle)\,d\boldsymbol{k}_1$$

$$+ \varepsilon^2\Big(\Omega_{\boldsymbol{k}}A^c_{\boldsymbol{k}} + \int F_{\boldsymbol{k}\boldsymbol{k}_1\boldsymbol{k}_2}(A^c_{\boldsymbol{k}_1}A^c_{\boldsymbol{k}_2}A^c_{\boldsymbol{k}_3=\boldsymbol{k}-\boldsymbol{k}_1-\boldsymbol{k}_2} + \langle A'_{\boldsymbol{k}_1}A'_{\boldsymbol{k}_2}\rangle A^c_{\boldsymbol{k}_3} + A^c_{\boldsymbol{k}_1}\langle A'_{\boldsymbol{k}_2}A'_{\boldsymbol{k}_3}\rangle$$

$$+ A^c_{\boldsymbol{k}_2}\langle A'_{\boldsymbol{k}_1}A'_{\boldsymbol{k}_3}\rangle + \langle A'_{\boldsymbol{k}_1}A'_{\boldsymbol{k}_2}A'_{\boldsymbol{k}_3}\rangle)\,d\boldsymbol{k}_1\,d\boldsymbol{k}_2\Big). \tag{4.2}$$

For the incoherent component of the amplitude $A'_{\boldsymbol{k}}$, we have

$$\frac{\partial A'_{\boldsymbol{k}}}{\partial t} = -\varepsilon \int H_{\boldsymbol{k}\boldsymbol{k}_1\boldsymbol{k}_2}(A^c_{\boldsymbol{k}_1}A'_{\boldsymbol{k}_2=\boldsymbol{k}-\boldsymbol{k}_1} + A'_{\boldsymbol{k}_1}A^c_{\boldsymbol{k}_2} + A'_{\boldsymbol{k}_1}A'_{\boldsymbol{k}_2} - \langle A'_{\boldsymbol{k}_1}A'_{\boldsymbol{k}_2}\rangle)\,d\boldsymbol{k}_1$$

$$+ \varepsilon^2\Big(\Omega_{\boldsymbol{k}}A'_{\boldsymbol{k}} + \int F_{\boldsymbol{k}\boldsymbol{k}_1\boldsymbol{k}_2}(A^c_{\boldsymbol{k}_1}A'_{\boldsymbol{k}_2}A^c_{\boldsymbol{k}_3=\boldsymbol{k}-\boldsymbol{k}_1-\boldsymbol{k}_2} + A'_{\boldsymbol{k}_1}A^c_{\boldsymbol{k}_2}A^c_{\boldsymbol{k}_3} + A^c_{\boldsymbol{k}_1}A^c_{\boldsymbol{k}_2}A'_{\boldsymbol{k}_3}$$

$$+ (A'_{\boldsymbol{k}_1}A'_{\boldsymbol{k}_2} - \langle A'_{\boldsymbol{k}_1}A'_{\boldsymbol{k}_2}\rangle)A^c_{\boldsymbol{k}_3} + A^c_{\boldsymbol{k}_1}(A'_{\boldsymbol{k}_2}A'_{\boldsymbol{k}_3} - \langle A'_{\boldsymbol{k}_2}A'_{\boldsymbol{k}_3}\rangle) + A^c_{\boldsymbol{k}_2}(A'_{\boldsymbol{k}_1}A'_{\boldsymbol{k}_3} - \langle A'_{\boldsymbol{k}_1}A'_{\boldsymbol{k}_3}\rangle)$$

$$+A'_{\boldsymbol{k}_1}A'_{\boldsymbol{k}_2}A'_{\boldsymbol{k}_3} - \langle A'_{\boldsymbol{k}_1}A'_{\boldsymbol{k}_2}A'_{\boldsymbol{k}_3}\rangle)\,d\boldsymbol{k}_1\,d\boldsymbol{k}_2\Big). \tag{4.3}$$

5. Moments of the Random Component in the Homogeneous Case. For the ensemble-averaged product of amplitudes of random fields in the homogeneous case, we obtain

$$\langle A'_{\boldsymbol{k}_1}A'_{\boldsymbol{k}_2}\rangle = \Gamma_2(\boldsymbol{k}_1)\delta(\boldsymbol{k}_1+\boldsymbol{k}_2), \qquad \langle A'_{\boldsymbol{k}_1}A'_{\boldsymbol{k}_2}A'_{\boldsymbol{k}_3}\rangle = \varepsilon\Gamma_3(\boldsymbol{k}_1,\boldsymbol{k}_2)\delta(\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3),$$

$$\langle A'_{\boldsymbol{k}_1}A'_{\boldsymbol{k}_2}A'_{\boldsymbol{k}_3}A'_{\boldsymbol{k}_4}\rangle = \Gamma_2(\boldsymbol{k}_1)\Gamma_2(\boldsymbol{k}_3)\delta(\boldsymbol{k}_1+\boldsymbol{k}_2)\delta(\boldsymbol{k}_3+\boldsymbol{k}_4) + \Gamma_2(\boldsymbol{k}_1)\Gamma_2(\boldsymbol{k}_2)\delta(\boldsymbol{k}_1+\boldsymbol{k}_3)\delta(\boldsymbol{k}_2+\boldsymbol{k}_4)$$

$$+ \Gamma_2(\boldsymbol{k}_1)\Gamma_2(\boldsymbol{k}_2)\delta(\boldsymbol{k}_1+\boldsymbol{k}_4)\delta(\boldsymbol{k}_2+\boldsymbol{k}_3) + \varepsilon\Gamma_4(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3)\delta(\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3+\boldsymbol{k}_4), \quad \text{etc.}$$

Equation (4.2) for the coherent component takes the form

$$\frac{\partial A^c_{\boldsymbol{k}}}{\partial t} = -\varepsilon \int H_{\boldsymbol{k}\boldsymbol{k}_1\boldsymbol{k}_2=\boldsymbol{k}-\boldsymbol{k}_1}A^c_{\boldsymbol{k}_1}A^c_{\boldsymbol{k}_2}\,d\boldsymbol{k}_1$$

$$+ \varepsilon^2\Big(\Omega_{\boldsymbol{k}} + \int (F_{\boldsymbol{k}\boldsymbol{k}_1-\boldsymbol{k}_1} + F_{\boldsymbol{k}\boldsymbol{k}\boldsymbol{k}_1} + F_{\boldsymbol{k}\boldsymbol{k}_1\boldsymbol{k}})\Gamma_2(\boldsymbol{k}_1)\,d\boldsymbol{k}_1\Big)A^c_{\boldsymbol{k}} + \varepsilon^2\int F_{\boldsymbol{k}\boldsymbol{k}_1\boldsymbol{k}_2}A^c_{\boldsymbol{k}_1}A^c_{\boldsymbol{k}_2}A^c_{\boldsymbol{k}_3=\boldsymbol{k}-\boldsymbol{k}_1-\boldsymbol{k}_2}\,d\boldsymbol{k}_1\,d\boldsymbol{k}_2. \tag{5.1}$$

The equations for the second- and third-order moments for stochastic fluctuations, which satisfy Eq. (4.3), are written as

$$\frac{\partial\Gamma_2(\boldsymbol{p})\delta(\boldsymbol{p}+\boldsymbol{q})}{\partial t} = -\varepsilon^2\delta(\boldsymbol{p}+\boldsymbol{q})\int (H_{-\boldsymbol{p},\boldsymbol{k}_1,-\boldsymbol{p}-\boldsymbol{k}_1}\Gamma_3(\boldsymbol{p},\boldsymbol{k}_1) + H_{\boldsymbol{p},\boldsymbol{k}_1,\boldsymbol{p}-\boldsymbol{k}_1}\Gamma_3(-\boldsymbol{p},\boldsymbol{k}_1))\,d\boldsymbol{k}_1$$

$$- \varepsilon((H_{\boldsymbol{q},\boldsymbol{q}+\boldsymbol{p},\boldsymbol{p}} + H_{\boldsymbol{q},-\boldsymbol{p},\boldsymbol{q}+\boldsymbol{p}})\Gamma_2(\boldsymbol{p}) + (H_{\boldsymbol{p},\boldsymbol{q}+\boldsymbol{p},\boldsymbol{q}} + H_{\boldsymbol{p},-\boldsymbol{q},\boldsymbol{q}+\boldsymbol{p}})\Gamma_2(\boldsymbol{q}))A^c_{\boldsymbol{q}+\boldsymbol{p}}$$

$$+ \varepsilon^2\Big(2\Omega_{\boldsymbol{p}}\Gamma_2(\boldsymbol{p})\delta(\boldsymbol{p}+\boldsymbol{q}) + \delta(\boldsymbol{p}+\boldsymbol{q})\int ((F_{\boldsymbol{p}\boldsymbol{p}\boldsymbol{k}_1} + F_{\boldsymbol{p}\boldsymbol{k}_1\boldsymbol{p}} + F_{\boldsymbol{p}\boldsymbol{k}_1-\boldsymbol{k}_1})\Gamma_2(-\boldsymbol{p})\Gamma_2(\boldsymbol{k}_1)$$

$$+ (F_{-\boldsymbol{q},-\boldsymbol{q},\boldsymbol{k}_1} + F_{-\boldsymbol{q},\boldsymbol{k}_1,-\boldsymbol{q}} + F_{-\boldsymbol{q},\boldsymbol{k}_1,-\boldsymbol{k}_1})\Gamma_2(\boldsymbol{p})\Gamma_2(\boldsymbol{k}_1))\,d\boldsymbol{k}_1\Big) + o(\varepsilon^2); \tag{5.2}$$

$$\frac{\partial\Gamma_3(\boldsymbol{p},\boldsymbol{q})}{\partial t} = -(H_{\boldsymbol{p},-\boldsymbol{q},\boldsymbol{p}+\boldsymbol{q}} + H_{\boldsymbol{p},\boldsymbol{p}+\boldsymbol{q},-\boldsymbol{q}})\Gamma_2(\boldsymbol{q})\Gamma_2(-\boldsymbol{p}-\boldsymbol{q})$$

$$- (H_{\boldsymbol{q},\boldsymbol{p}+\boldsymbol{q},-\boldsymbol{p}} + H_{\boldsymbol{q},-\boldsymbol{p},\boldsymbol{q}+\boldsymbol{p}})\Gamma_2(-\boldsymbol{p}-\boldsymbol{q})\Gamma_2(\boldsymbol{p})$$

$$- (H_{-\boldsymbol{p}-\boldsymbol{q},-\boldsymbol{p},-\boldsymbol{q}} + H_{-\boldsymbol{p}-\boldsymbol{q},-\boldsymbol{q},-\boldsymbol{p}})\Gamma_2(\boldsymbol{p})\Gamma_2(\boldsymbol{q}) + O(\varepsilon). \tag{5.3}$$

In the rigorous sense, the expressions for the correlations require certain generalization, since the term proportional to the amplitude of coherent fluctuations in equations for the stochastic component, in the simplest case, may contain several δ functions corresponding to a discrete set of wave vectors in three-wave resonance. Nevertheless, assuming that these wave vectors are located near the origin of the space of wavenumbers and assuming, in a certain approximation, that the difference in wave vectors is insignificant for the stochastic component, we may consider them as exact. As in [10], we obtain equations of the kinetic type for the stochastic component.

Taking into account that the system contains a small parameter, following [9], we expand the sought quantities into series in ε:

$$A^c_{\boldsymbol{k}} = A^{c(0)}_{\boldsymbol{k}} + \varepsilon A^{c(1)}_{\boldsymbol{k}} + \varepsilon^2 A^{c(1)}_{\boldsymbol{k}} + o(\varepsilon^2), \qquad \Gamma_2(\boldsymbol{k}) = \Gamma_2^{(0)}(\boldsymbol{k}) + \varepsilon\Gamma_2^{(1)}(\boldsymbol{k}) + O(\varepsilon^2),$$

$$\Gamma_3(\boldsymbol{k},\boldsymbol{k}_1) = \Gamma_3^{(0)}(\boldsymbol{k},\boldsymbol{k}_1) + \varepsilon\Gamma_3^{(1)}(\boldsymbol{k},\boldsymbol{k}_1) + O(\varepsilon^2), \quad \Gamma_4(\boldsymbol{k},\boldsymbol{k}_1,\boldsymbol{k}_2) = \Gamma_4^{(0)}(\boldsymbol{k},\boldsymbol{k}_1,\boldsymbol{k}_2) + O(\varepsilon), \tag{5.4}$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial t_0} + \varepsilon\,\frac{\partial}{\partial t_1} + \varepsilon^2\,\frac{\partial}{\partial t_2} + o(\varepsilon^2).$$

Substituting expansions (5.4) into system (5.1)–(5.3), we find that the amplitudes of the coherent structure are independent of t_0, and the function $\Gamma_2[\boldsymbol{k}]$ is independent of t_0 and t_1. Elimination of secular terms yields the following equations:

— dynamic equation for the coherent part

$$\frac{\partial A^{c(0)}_{\boldsymbol{k}}}{\partial t_1} = \lim_{t_0\to\infty}\frac{1}{t_0}\int\limits_0^{t_0} R\,dt_0 \tag{5.5}$$

[R is the right side of the equation for the amplitudes $A^{c(0)}_{\boldsymbol{k}}$];

— equation of the kinetic type for the incoherent part

$$\frac{\partial\Gamma_2^{(0)}(\boldsymbol{p})}{\partial t_2} = \lim_{t_1\to\infty}\frac{1}{t_1}\int\limits_0^{t_1}\lim_{t_0\to\infty}\frac{1}{t_0}\int\limits_0^{t_0} R_1\,dt_0\,dt_1$$

(R_1 is the right side of the equation for $\Gamma_2^{(0)}(\boldsymbol{p})$).

In the equation for the incoherent component, one should take into account the mean part of the nonstationary system of equations that describe the coherent component in the scale t_1.

6. Some Possible Solutions of the System for the Coherent Component and Stochastic Fluctuations. For the three-wave resonance, in the simplest case, we have

$$A^c_{\boldsymbol{k}} = b\delta(\boldsymbol{k}) + (a_1\delta(\boldsymbol{k}-\boldsymbol{k}_1^{(0)}) + a_2\delta(\boldsymbol{k}-\boldsymbol{k}_2^{(0)}) + a_3\delta(\boldsymbol{k}-\boldsymbol{k}_3^{(0)}) + \text{c.c.}),$$

where c.c. is the sum of complex-conjugate terms, $\boldsymbol{k}_1^{(0)}$, $\boldsymbol{k}_2^{(0)}$, $\boldsymbol{k}_3^{(0)}$ is one possible set of vectors that satisfy the three-wave resonance equations (see, e.g., [9]).

A numerical investigation yields the following pattern of the three-wave resonance for one of the least decaying modes of the Tollmien–Schlichting waves for the velocity profile in the turbulent boundary layer [11] (see Fig. 2, where α and β are the longitudinal and transverse components of the wave vector, respectively.)

The vectors are chosen to satisfy the equality $\boldsymbol{k}_1 + \boldsymbol{k}_2 = \boldsymbol{k}$, and the ends of the vectors should lie on the resonance curves determined by the vector $\boldsymbol{k}$. In this case, Eqs. (5.5) for the amplitudes of the coherent component can be represented as

$$\frac{da_1}{dt} = \varepsilon[a_2a_3(H_{\boldsymbol{k}_1^{(0)}\boldsymbol{k}_2^{(0)}\boldsymbol{k}_3^{(0)}} + H_{\boldsymbol{k}_1^{(0)}\boldsymbol{k}_3^{(0)}\boldsymbol{k}_2^{(0)}})] + \varepsilon^2(Q_1),$$

$$\frac{da_2}{dt} = \varepsilon[a_1a_3^*(H_{\boldsymbol{k}_2^{(0)}\boldsymbol{k}_1^{(0)}-\boldsymbol{k}_3^{(0)}} + H_{\boldsymbol{k}_2^{(0)}-\boldsymbol{k}_3^{(0)}\boldsymbol{k}_1^{(0)}})] + \varepsilon^2(Q_2),$$

$$\frac{da_3}{dt} = \varepsilon[a_1a_2^*(H_{\boldsymbol{k}_3^{(0)}\boldsymbol{k}_1^{(0)}-\boldsymbol{k}_2^{(0)}} + H_{\boldsymbol{k}_3^{(0)}-\boldsymbol{k}_2^{(0)}\boldsymbol{k}_1^{(0)}})] + \varepsilon^2(Q_3),$$

where Q_i ($i = 1, 2, 3$) is the set of linear and cubic terms with respect to the amplitude; the superscript asterisk denotes complex conjugation.

7. Continuum Analog of Equations for the Three-Wave Resonance Amplitudes. An infinite number of pairs of wave vectors satisfy the condition of three-wave resonance simultaneously; therefore, we consider all these pairs. The solution $A^c_{\boldsymbol{k}}$ can be sought in the form of an infinite sum of δ functions, as was done in Sec. 6.

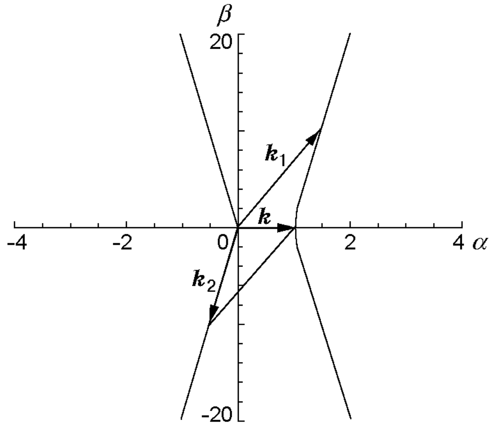

Fig. 2. Three-wave resonance on the profile of the turbulent boundary layer for one of the least decaying modes of the Tollmien–Schlichting waves.

This allows one to obtain equations for the amplitudes of pairs of wave vectors, which are continuous functions of the point on the resonance curve. Finally, we obtain integrodifferential equations of the form

$$\frac{\partial \alpha_1}{\partial t} = \varepsilon \int\limits_L \alpha_2(l)\alpha_3(l)h_1(l)\,dl + \ldots, \qquad \frac{\partial \alpha_2(l)}{\partial t} = \varepsilon\alpha_1\alpha_3^*(l)h_2(l) + \ldots, \qquad \frac{\partial \alpha_3(l)}{\partial t} = \varepsilon\alpha_1\alpha_2^*(l)h_3(l) + \ldots,$$

where l is the parameter along the resonance curve L, $h_i(l)$ $(i = 1, 2, 3)$ are certain combinations of matrix elements, and $\alpha_i(l)$ $(i = 1,\ 2,\ 3)$ are the amplitudes of waves in the state of three-wave resonance.

8. Correlation Function. In the general case, the correlation functions should also be sought in the form of sums of δ functions (it follows from the fact that the right and left sides of equations for correlation functions should have the corresponding elements). Then, in the presence of a coherent structure, we have

$$\langle A'_{\boldsymbol{k}_1} A'_{\boldsymbol{k}_2}\rangle = \sum_i \Gamma_2^{(i)}(\boldsymbol{k}_1)\delta(\boldsymbol{k}_1 + \boldsymbol{k}_2 - \boldsymbol{k}_i^{(0)}),$$

$$\langle A'_{\boldsymbol{k}_1} A'_{\boldsymbol{k}_2} A'_{\boldsymbol{k}_3}\rangle = \varepsilon \sum_i \Gamma_3^{(i)}(\boldsymbol{k}_1, \boldsymbol{k}_2)\delta(\boldsymbol{k}_1 + \boldsymbol{k}_2 + \boldsymbol{k}_3 - \boldsymbol{k}_i^{(0)}),$$

$$\langle A'_{\boldsymbol{k}_1} A'_{\boldsymbol{k}_2} A'_{\boldsymbol{k}_3} A'_{\boldsymbol{k}_4}\rangle = \sum_{i,j} \Gamma_2^{(i)}(\boldsymbol{k}_1)\Gamma_2^{(j)}(\boldsymbol{k}_3)\delta(\boldsymbol{k}_1 + \boldsymbol{k}_2 - \boldsymbol{k}_i^{(0)})\delta(\boldsymbol{k}_3 + \boldsymbol{k}_4 - \boldsymbol{k}_j^{(0)})$$

$$+ \sum_{i,j} \Gamma_2^{(i)}(\boldsymbol{k}_1)\Gamma_2^{(j)}(\boldsymbol{k}_2)\delta(\boldsymbol{k}_1 + \boldsymbol{k}_3 - \boldsymbol{k}_i^{(0)})\delta(\boldsymbol{k}_2 + \boldsymbol{k}_4 - \boldsymbol{k}_j^{(0)})$$

$$+ \sum_{i,j} \Gamma_2^{(i)}(\boldsymbol{k}_1)\Gamma_2^{(j)}(\boldsymbol{k}_2)\delta(\boldsymbol{k}_1 + \boldsymbol{k}_4 - \boldsymbol{k}_i^{(0)})\delta(\boldsymbol{k}_2 + \boldsymbol{k}_3 - \boldsymbol{k}_j^{(0)}) + \varepsilon \sum_i \Gamma_4^{(i)}(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3)\delta(\boldsymbol{k}_1 + \boldsymbol{k}_2 + \boldsymbol{k}_3 + \boldsymbol{k}_4 - \boldsymbol{k}_i^{(0)}).$$

Hence, the two-point spatial correlator acquires the form

$$\langle A'(\boldsymbol{r}_1)A'(\boldsymbol{r}_2)\rangle = \Big\langle \iint A'(\boldsymbol{k}_1)\,\mathrm{e}^{i\boldsymbol{k}_1\boldsymbol{r}_1}\ d\boldsymbol{k}_1\, A'(\boldsymbol{k}_2)\,\mathrm{e}^{i\boldsymbol{k}_2\boldsymbol{r}_2}\ d\boldsymbol{k}_2 \Big\rangle = \iint \langle A'(\boldsymbol{k}_1)A'(\boldsymbol{k}_2)\rangle\,\mathrm{e}^{i\boldsymbol{k}_1\boldsymbol{r}_1}\,\mathrm{e}^{i\boldsymbol{k}_2\boldsymbol{r}_2}\ d\boldsymbol{k}_1\,d\boldsymbol{k}_2$$

$$= \iint \sum_i \Gamma^{(i)}(\boldsymbol{k}_1)\delta(\boldsymbol{k}_1 + \boldsymbol{k}_2 - \boldsymbol{k}_i^{(0)})\,\mathrm{e}^{i\boldsymbol{k}_1\boldsymbol{r}_1}\,\mathrm{e}^{i\boldsymbol{k}_2\boldsymbol{r}_2}\ d\boldsymbol{k}_1\,d\boldsymbol{k}_2 = \int \Big(\sum_i \Gamma^{(i)}(\boldsymbol{k}_1)\,\mathrm{e}^{i\boldsymbol{k}_i^{(0)}\boldsymbol{r}_2}\Big)\,\mathrm{e}^{i\boldsymbol{k}_1(\boldsymbol{r}_1-\boldsymbol{r}_2)}\ d\boldsymbol{k}_1.$$

This expression is a natural generalization of spatial homogeneity in the presence of coherent structures. As it could be expected, in the case of small $\boldsymbol{k}_i^{(0)}$, the expression reduces to an ordinary homogeneous correlator. For large values of $\boldsymbol{r}_2$, however, the arguments of the expression $\mathrm{e}^{\boldsymbol{k}_i^{(0)}\boldsymbol{r}_2}$ are not small. Therefore, in the general case, the correlator is either a periodic function in the space or differs from zero in a finite region of the space.

Thus, the presence of internal resonance in the turbulent boundary layer, even in the case of excitation of an infinite number of degrees of freedom, allows one to reveal some part of the field of fluctuations, which may be considered as a dynamic coherent component. It should be noted that, in the general case, this structure has a continuous spectrum due to the multiple three-wave resonance, i.e., possibility of simultaneous resonances of an infinite set of wave triads with wave vectors that satisfy the condition of three-wave resonance. Because of that, it seems to be impossible to identify this component on the background of the field of fluctuations.

The presence of the coherent structure leads to natural generalization of the notion of statistical homogeneity; as a result, the two-point correlator of the field of fluctuations becomes a periodic function of spatial coordinates. Note, the method of moments yields equations for the stochastic component, which are more generic than the kinetic equation in the three-wave resonance approximation.

This work was supported by the Russian Foundation for Fundamental Research (Grant Nos. 99-01-01239 and 00-15-96069).

REFERENCES

1. J. Boussinesq, "Essai sur la thé des caux courantes," *Mem. Présentés Divers Savants Acad. Sci.*, **23**, 46 (1877).
2. L. Prandtl, "Uber die ausgebildete Turbulenz," *Z. Angew. Math. Mech.*, **5**, 136–139 (1925).
3. J. Smagorinsky, "General circulation experiments with the primitive equations," *Mon. Weather Rev.*, **91**, 99–164 (1963).
4. Y. Zhou and G. Vahala, "Reformulation of recursive-renormalization-group-based subgrid modeling of turbulence," *Phys. Rev. E*, **47**, No. 4, 2503–2519 (1993).
5. Branko Kosovic, "Subgrid-scale modeling for the large-eddy simulation of high Reynolds-number boundary layers," *J. Fluid Mech.*, **336**, 151–182 (1997).
6. L. D. Landau and E. M. Lifshits, *Course of Theoretical Physics*, Vol. 6: *Fluid Mechanics*, Pergamon Press, Oxford-Elmsford, New York (1987).
7. B. J. Cantwell, "Organized motion in turbulent flow," *Annu. Rev. Fluid Mech.*, **13**, 457–515 (1981).
8. A. K. M. F. Hussain, "Coherent structure — reality and myth," *Phys. Fluids*, **26**, No. 10, 2816–2863 (1983).
9. R. C. Davidson, *Method in Nonlinear Plasma Theory*, Academic Press, New York–London (1972). (*Pure Appl. Phys.*, **37**.)
10. V. A. Zharov, "Wave structure of a developed turbulent boundary layer," *Uch. Zap. TsAGI*, **17**, No. 5, 28–38 (1986).
11. I. G. Dodonov, V. A. Zharov, and Yu. I. Khlopkov, "Localized coherent structures in the boundary layer," *J. Appl. Mech. Tech. Phys.*, **41**, No. 6, 1012–1019 (2000).
12. D. J. Benney and R. F. Bergeron, "A new class of nonlinear waves in parallel flows," *Stud. Appl. Math.*, **48**, No. 3, 181–204 (1969).
13. H. Solwen and C. E. Grosh, "The continuous spectrum of Orr–Sommerfeld equation. Part 2. Eigenfunction expansion," *J. Fluid Mech.*, **104**, Part 1, 445–465 (1981).